

**On the equation  $x(x+d)\cdots(x+(k-1)d) = y(y+d)\cdots(y+(mk-1)d)$** 

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## §1. INTRODUCTION

Throughout this paper, the letters  $d, k, m, x, y$  denote positive integers such that  $k \geq 2$ ,  $m \geq 2$  and we write  $K, M$  for positive numbers given by

$$(1) \quad K^2 = k \log k, \quad M^2 = m(m-1)/2.$$

We consider the equation

$$(2) \quad x(x+d)\cdots(x+(k-1)d) = y(y+d)\cdots(y+(mk-1)d).$$

It is proved in [2] that equation (2) with  $d=1$  implies that  $\max(x, y, k)$  is bounded by an effectively computable number depending only on  $m$ . Now, we extend this result for any fixed  $d$ . See Corollary 2 which is an immediate consequence of the following two more general results.

**THEOREM 1.** *Let  $\varepsilon > 0$ . There exists an effectively computable number  $C$  depending only on  $m$  and  $\varepsilon$  such that equation (2) with  $\max(x, y, k) \geq C$  implies that*

$$(3) \quad d \geq y^{(1-\varepsilon)/(m+1)}.$$

Next, we combine Theorem 1 and Corollary 3 for deriving the following result.

COROLLARY 1. *Let  $\varepsilon > 0$ . There exists an effectively computable number  $C_1$  depending only on  $m$  and  $\varepsilon$  such that equation (2) with  $\max(x, y, k) \geq C_1$  implies that*

$$(4) \quad \log d \geq \left( \frac{M}{m(m+1)} - \varepsilon \right) K$$

where  $K$  and  $M$  are given by (1).

As already mentioned, the following result is an immediate consequence of Theorem 1 and Corollary 1.

COROLLARY 2. *The equation (2) implies that  $\max(x, y, k)$  is bounded by an effectively computable number depending only on  $m$  and  $d$ .*

## §2. A $p$ -ADIC ARGUMENT

We extend Lemma 4 of [2] by proving the following result.

LEMMA 1. *Let  $\varepsilon > 0$ . There exists an effectively computable number  $C_2$  depending only on  $\varepsilon$  such that equation (2) with  $k \geq C_2$  implies that*

$$(5) \quad \log x \geq (1 - \varepsilon) MK.$$

PROOF. By (2), we observe that

$$(6) \quad kd \leq x.$$

Further, we notice from (2) and (6) that

$$(mk - 1)! d^{mk-1} \leq k! x^k$$

which sharpens (6) as

$$(7) \quad d < x^{k/(mk-1)}.$$

We may assume that  $k$  exceeds a sufficiently large effectively computable number depending only on  $\varepsilon$ . If  $d$  is divisible by all the primes not exceeding  $MK/m$ , then we observe from (7) that

$$k \log x \geq (mk - 1) \sum_{p \leq MK/m} \log p$$

and (5) follows from Prime Number Theorem. Thus, we may suppose that there exists a prime  $p$  satisfying

$$(8) \quad p \leq MK/m \quad \text{and} \quad p \nmid d.$$

Let  $i_0$  with  $0 \leq i_0 < k$  satisfy

$$\text{ord}_p(x + i_0 d) = \max_{0 \leq i < k} \text{ord}_p(x + id).$$

We count the power of  $p$  in the factorisations of both the sides of (2). By (8), the power of  $p$  in the factorisation of the left hand side of (2) is at most

$$\text{ord}_p(x + i_0 d) + \text{ord}_p((k-1)!) \leq \frac{\log(x + (k-1)d)}{\log p} + \frac{k}{p-1}$$

and the power of  $p$  occurring in the right hand side of (2) is at least

$$\left\lfloor \frac{mk}{p} \right\rfloor + \left\lfloor \frac{mk}{p^2} \right\rfloor > \frac{k}{p} \left( m + \frac{m}{p} \right) - 2.$$

Therefore

$$\frac{k}{p} (m-1) \leq \frac{k}{p} \left( m + \frac{m}{p} - \frac{p}{p-1} \right) < \frac{\log(x + (k-1)d)}{\log p} + 2$$

which, together with (8) and (1), implies that

$$\log(x + (k-1)d) \geq (1 - \varepsilon/2)MK.$$

On the other hand, we observe from (6) that

$$\log(x + (k-1)d) \leq \log x + \log 2.$$

Finally, we combine the above estimates for  $\log(x + (k-1)d)$  to conclude (5).  $\square$

Before we close this section, we derive the following lower bound for  $y + ((mk-1)/2)d$  from Lemma 1.

**COROLLARY 3.** *Let  $\varepsilon > 0$ . If (2) holds, then*

$$(9) \quad \log \left( y + \left( \frac{mk-1}{2} \right) d \right) \geq (1 - \varepsilon)MK/m \quad \text{for } k \geq C_2.$$

**PROOF.** By the arithmetic-geometric mean, we observe from (2) that

$$(10) \quad x < \left( y + \left( \frac{mk-1}{2} \right) d \right)^m.$$

Now, we combine (10) and (5) to conclude (9).  $\square$

### §3. PROOF OF THEOREM 1

Let  $0 < \varepsilon < 1$ . We assume (2) with

$$(11) \quad d < y^{(1-\varepsilon)/(m+1)}.$$

Then, we observe that  $x > y > d$ . Now, by writing  $x = x_1 + d$  and  $y = y_1 + d$ , there is no loss of generality in assuming that

$$(12) \quad (x+d) \cdots (x+kd) = (y+d) \cdots (y+mkd),$$

instead of (2). We denote by  $c_1, \dots, c_5$  effectively computable positive numbers

depending only on  $m$  and  $\varepsilon$ . We may assume that  $y \geq c_1$  with  $c_1$  sufficiently large, otherwise (11), Corollary 3, (1) and (12) imply that  $\max(x, y, k) \leq c_2$ . Further, we apply Corollary 3 for deriving that

$$(13) \quad \log(y + (mk - 1)d) \geq c_3 K.$$

Let  $A_j(m, k)$ ,  $B_j = B_j(m, k)$  and  $H_j(m, k)$  be given by (2)–(5) of [2]. Then

$$(14) \quad (z + d) \cdots (z + mkd) = \sum_{j=0}^{mk} A_j(m, k) d^j z^{mk-j}$$

and

$$(15) \quad (z^m + B_1 dz^{m-1} + \cdots + B_m d^m)^k = \sum_{j=0}^{mk} H_j(m, k) d^j z^{mk-j}.$$

We write

$$F_d(x, k) = (x + d) \cdots (x + kd), \quad F_d(y, m, k) = (y + d) \cdots (y + mkd)$$

and

$$(16) \quad \Lambda_d = \Lambda_d(y, m, k) = y^m + B_1 dy^{m-1} + \cdots + B_m d^m.$$

We observe that  $F_1(x, k) = F(x, k)$ ,  $F_1(y, m, k) = F(y, m, k)$  and  $\Lambda_1 = \Lambda$  where  $F(x, k)$ ,  $F(y, m, k)$  and  $\Lambda$  are given by (28) and (29) of [2]. We apply arithmetic-geometric mean to the left hand side of (12) to obtain

$$(17) \quad F_d(x, k) < \left( x + \frac{k+1}{2} d \right)^k.$$

Let  $f = -(k+1)/2$  and  $a_i(f, k)$  with  $1 \leq i \leq k$  be given by (44) and (45) of [2].

By applying (14) and (15), we argue as in the proof of [2, Lemma 5] for obtaining the following extension.

$$(18) \quad F_d(y, m, k) < (\Lambda_d + (4k^{2m-1})^{-1})^k,$$

$$(19) \quad F_d(y, m, k) > (\Lambda_d - (2k^{2m-1})^{-1})^k$$

and

$$(20) \quad F_d(x, k) > \left( x + \frac{k+1}{2} d - (4k^{2m-1})^{-1} \right)^k.$$

As in the proof of [2, Theorem 2], we derive from (12), (17), (18), (19), (20), (16) and [2, Lemma 3] that

$$(21) \quad x = \Lambda_d + fd.$$

By substituting (21) in the left hand side of (12), we derive that

$$(22) \quad F_d(x, k) = \Lambda_d^k + a_2(f, k) d^2 \Lambda_d^{k-2} + \cdots + a_k(f, k) d^k.$$

By (22) and (16), we obtain that

$$F_d(x, k) = \sum_{j=0}^{mk} T_{j,d}(m, k) d^j y^{mk-j}$$

where

$$T_{j,d}(m,k) = \begin{cases} H_j(m,k) & \text{for } 0 \leq j < 2m, \\ H_j(m,k) + a_2(f,k)d^{-2(m-1)}H_{j-2m}(m,k-2) + \dots \\ \quad + a_h(f,k)d^{-h(m-1)}H_{j-hm}(m,k-h) & \text{for } hm \leq j < (h+1)m \text{ and } 2 \leq h < k, \\ B_m^k + a_2(f,k)d^{-2(m-1)}B_m^{k-2} + \dots + a_k(f,k)d^{-k(m-1)} & \text{for } j = mk. \end{cases}$$

Now, we follow the proof of (57) and (58) of [2] to derive that

$$(23) \quad H_j(m,k) = A_j(m,k) \quad \text{for } 0 \leq j < 2m,$$

$$(24) \quad (H_{2m}(m,k) - A_{2m}(m,k))d^{2m} = \frac{k(k+1)(k-1)}{24}d^2.$$

It is easy to calculate

$$H_4(2,k) - A_4(2,k) = (4k^5 - 5k^3 + k)/90$$

which contradicts (24) with  $m=2$ . Thus, we derive that  $m > 2$ .

We apply a result of Balasubramanian (see [2, Appendix]) to derive from (23) and  $m > 2$  that

$$(25) \quad k \leq c_4.$$

By [2, Lemma 3], we observe that the absolute value of the left hand side of (24) is at least  $d^{2m}/k^{4m-1}$ . Therefore, we obtain from (24) that  $24d^{2m-2} \leq k^{4m+2}$  which, together with (25), implies that

$$(26) \quad d \leq c_5.$$

By (25) and [2, Lemma 9], we may assume that  $d \geq 2$ . We write

$$L(X,Y) = (X+d) \cdots (X+kd) - (Y+d) \cdots (Y+mkd)$$

and

$$\ell(Y) = L\left(Y^m + B_1dY^{m-1} + \dots + B_md^m - \frac{k+1}{2}d,Y\right).$$

As in the proof of [2, Lemma 9], we derive from (21), (25) and (26) that

$$(27) \quad \ell(Y) \equiv 0$$

and

$$(28) \quad -L(0,Y) = (Y+d) \cdots (Y+mkd) - k!d^k$$

is reducible over the field of rational numbers. Thus, by Gauss Lemma, there exist monic polynomials  $f(Y)$  and  $g(Y)$  with integral coefficients such that

$$(29) \quad -L(0,Y) = f(Y)g(Y).$$

Furthermore, we may assume that the degree  $\nu$  of  $f(Y)$  satisfies

$$(30) \quad [(mk+1)/2] \leq \nu < mk.$$

Now, we shall apply [1, Theorem 1], due to Tatuzawa, to  $f(Y)$  at the points  $-(v-i+1)d =: y_i$  with  $0 \leq i \leq v$ . By (29), (28) and (30), we observe that

$$(31) \quad |f(y_i)| \leq |L(0, y_i)| = k! d^k \quad \text{for } 0 \leq i \leq v.$$

On the other hand, we derive from [1, Theorem 1] that

$$(32) \quad \max_{0 \leq i \leq v} |f(y_i)| \geq v! (d/2)^v.$$

Since the right hand side of (32) is an increasing function of  $v$ , we derive from (31), (32) and (30) that

$$k! d^k \geq [(mk+1)/2]! (d/2)^{[(mk+1)/2]}.$$

This, since  $d \geq 2$  and  $m > 2$ , implies that  $m = 3$ ,  $k = 2$ ,  $d = 2$ . Then, by comparing the constant term in (27), we obtain

$$B_3^2(3, 2) - \frac{1}{64} = 6!$$

which is not possible, since  $B_3(3, 2)$  is a rational number.  $\square$

#### §4. PROOF OF COROLLARY 1

We denote by  $c_6, \dots, c_{10}$  effectively computable positive numbers depending only on  $m$  and  $\varepsilon$ . Let  $k \leq c_6$ . Then, we may assume that  $d \leq c_7$ , otherwise (4) follows immediately. Now, we refer to (3) and (2) for concluding that  $\max(x, y, k) \leq c_8$ . Therefore, we may assume that  $k > c_9$  with  $c_9$  sufficiently large. Thus, the assertion (9) of Corollary 3 is valid. Next, we observe that (9) implies (4) whenever  $d > y$ . Therefore, we may also suppose that  $d \leq y$ .

Let  $0 < \varepsilon_1 < 1$  to be suitably chosen depending only on  $m$  and  $\varepsilon$ . By (9) and  $d \leq y$ , we have

$$(33) \quad \log y \geq \left( \frac{M}{m} - \varepsilon_1 \right) K.$$

Further, we apply Theorem 1 for deriving that

$$(34) \quad \log d \geq \left( \frac{1}{m+1} - \varepsilon_1 \right) \log y.$$

Now, we combine (34) and (33) to obtain

$$(35) \quad \log d \geq \left( \frac{M}{m(m+1)} - c_{10} \varepsilon_1 \right) K.$$

Finally, we take  $\varepsilon_1 = \varepsilon/c_{10}$  to obtain (4) from (35).  $\square$

#### REFERENCES

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